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LIE GROUPS AND TEICHMÜLLER SPACE

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§1. INTRODUCTION

It is well-known that Teichmüller space is a ball of dimension $(6g - 6)$. What is perhaps less well-known is that this result can be interpreted purely in terms of the fundamental group $\pi_1(\Sigma)$ of a compact surface Σ and its representations in $PSL(2, \mathbb{R})$. Every conformal structure on Σ can be uniformized, thereby representing Σ as the quotient of the upper half-plane by a subgroup of $PSL(2, \mathbb{R})$ isomorphic to $\pi_1(\Sigma)$. This describes a homomorphism from $\pi_1(\Sigma)$ to $PSL(2, \mathbb{R})$ well-defined up to conjugation by an element of $PSL(2, \mathbb{R})$. It can be proved (see [4], [6]) that the set of all such homomorphisms constitutes a connected component of the topological space $\mathrm{Hom}(\pi_1(\Sigma); PSL(2, \mathbb{R}))/PSL(2, \mathbb{R})$. We conclude, therefore, that there is a component of this space homeomorphic to a Euclidean space $\mathbb{R}^{(6g-6)}$.

In this paper we shall demonstrate the existence of an analogous component—which we shall call the *Teichmüller* component—where we replace the group $PSL(2, \mathbb{R})$ by $PSL(n, \mathbb{R})$ or more generally by the adjoint group of a split real form G^r of any complex simple Lie group G^c (recall that the split real forms of the classical groups are $SL(n, \mathbb{R})$, $SO(n+1, n)$, $Sp(2n, \mathbb{R})$ and $SO(n, n)$). This component is homeomorphic to $\mathbb{R}^{(2g-2)\dim G^r}$ and contains Teichmüller space in the sense that there is a distinguished three-dimensional subgroup $PSL(2, \mathbb{R}) \subset G^r$ which embeds the uniformizing representations of $\pi_1(\Sigma)$ as a subspace of the Teichmüller component. We are therefore considering an extension of ordinary Teichmüller space.

To be more precise, we have:

Theorem A. *Let Σ be a compact oriented surface of genus $g > 1$ and let G^r be the adjoint group of the split real form of a complex simple Lie group G^c. Let $Hom^+(\pi_1(\Sigma); G^r)$ denote the space of homomorphisms from the fundamental group to G^r which act completely reducibly on the Lie algebra of G^r. Then the quotient of $Hom^+(\pi_1(\Sigma); G^r)$ by the conjugation action of G^r has a connected component homeomorphic to a Euclidean space of dimension $(2g-2)\dim G^r$.*

Returning to the case of $PSL(2, \mathbb{R})$, there is an easy way to identify this distinguished component. Every homomorphism from $\pi_1(\Sigma)$ to $PSL(2, \mathbb{R})$ defines an associated (flat) principal $PSL(2, \mathbb{R})$- bundle which is topologically classified by its reduction to a maximal compact subgroup—in this case the circle. The first Chern class of this circle bundle then provides an integer invariant and it turns out that Teichmüller space is precisely the subspace for which this integer takes the value $(2g - 2)$. (In fact an outer automorphism of $PSL(2, \mathbb{R})$ yields an isomorphic copy of the space corresponding to the value $-(2g-2)$ so that in truth there are *two* components homeomorphic to a ball). In the latter half of this paper we consider whether this feature holds for the Teichmüller component of $PSL(n, \mathbb{R})$ if

$n > 2$. Here the maximal compact subgroup is $PSO(n)$ and principal bundles with this structure group are topologically classified by 2-torsion classes and not integers. We find in this case that the Teichmüller component does *not* fill up the space of homomorphisms in a particular topological class of principal bundles. There is another component which contains homomorphisms to the compact group $PSO(n)$. (As above for n even an isomorphic copy of the Teichmüller component is created by an outer automorphism of $PSL(n, \mathbb{R})$). Roughly speaking, a homomorphism from $\pi_1(\Sigma)$ to $PSL(n, \mathbb{R})$ can be connected through a path either to a homomorphism to the projective orthogonal group, or to a homomorphism arising from a uniformization.

This information about the components can be summarized in the following:

THEOREM B. *The space* $\mathrm{Hom}^+(\pi_1(\Sigma); PSL(n, \mathbb{R}))/PSL(n, \mathbb{R})$ *has, for* $n > 2$, *three connected components if* n *is odd and six components if* n *is even.*

The method we use for proving the two theorems above is an analytical one—the theory of *Higgs bundles* developed by the author, C. Simpson, K. Corlette, S. K. Donaldson and others. A homomorphism from $\pi_1(\Sigma)$ to G^c defines a flat principal G^c-bundle. Given a conformal structure on Σ, a theorem of Corlette and Donaldson associates a natural G-connection A, where G is the maximal compact subgroup of G^c, and a Higgs field $\Phi \in H^0(\Sigma; \mathrm{ad}\, P \otimes K)$ which together satisfy the equations $F_A + [\Phi, \Phi^*] = 0$. Solutions to these equations can in turn be described by the holomorphic geometry of the principal bundle and Higgs field, using theorems of the author and Simpson. This provides a holomorphic parametrization of the equivalence classes of homomorphisms from $\pi_1(\Sigma)$ to G^c, which can easily be adapted to reality conditions.

To prove Theorem A for an arbitrary simple Lie group we make essential use of the results of Kostant on the principal 3-dimensional subgroup. The split real form of this gives the homomorphism from $PSL(2, \mathbb{R})$ to G^r which embeds Teichmüller space in the Teichmüller component.

To prove Theorem B we use the L^2-norm squared of the Higgs field Φ. For general reasons this is a proper non-negative function and thus on each component of the space of equivalence classes of homomorphisms has a minimum. By using the holomorphic point of view and some boot-strapping induction the minima can be calculated.

Unfortunately, the analytical point of view used for the proofs gives no indication of the *geometrical* significance of the Teichmüller component. Teichmüller space itself is, as we know, not simply a space of homomorphisms of a fundamental group, but more importantly is the space of conformal structures (or metrics of constant negative curvature) modulo the action of *diffeomorphisms* homotopic to the identity. This geometrical interpretation gives a natural action of the mapping class group on Teichmüller space. We do have an action in the general case too, but there is no geometrical interpretation to support it. The action on the moduli space of homomorphisms exists via the outer automorphisms of $\pi_1(\Sigma)$, and this at most permutes components: on the other hand since Teichmüller space is embedded in the Teichmüller component, the whole component is preserved by the action. This provokes consideration of the quotient of the Teichmüller component by the mapping class group and possible compactifications, but without further geometrical information it is difficult to proceed.

There is, despite this lack of information, one example which supports the view that the generalized Teichmüller spaces introduced here parametrize geometric structures on the surface Σ. In [5], Goldman constructs a Teichmüller space for convex $\mathbb{R}P^2$ structures on a surface. These are essentially uniformizations of Σ by a convex open set in the real projective plane acted upon by the group of real projective transformations. Goldman's space is an open contractible subspace of our Teichmüller component for $PSL(3, \mathbb{R})$ and is quite possibly the whole space. The general case of $PSL(n, \mathbb{R})$, however, remains obscure.

§2. HIGGS BUNDLES

We recall here, for the reader's benefit, the basic facts concerning Higgs bundles which we shall be using. Details may be found in [2], [3], [6], [7], [12], [13] and [14].

Let Σ be a compact Riemann surface of genus g and V a holomorphic vector bundle over Σ. A *Higgs bundle* is a pair (V, Φ) where Φ is a holomorphic section of End $V \otimes K$, K being the canonical line bundle over Σ. The section Φ is called a Higgs field.

A Higgs bundle is said to be *stable* if for each subbundle $U \subset V$ for which $\Phi(U) \subseteq U \otimes K$ (considering Φ as a map from V to $V \otimes K$),

$$\frac{\deg U}{\operatorname{rk} U} < \frac{\deg V}{\operatorname{rk} V}.$$

Stability exhibits the following properties:

(1) If α is a holomorphic automorphism of V and (V, Φ) is stable, then $(V, \alpha^* \Phi)$ is also stable.
(2) If (V, Φ) is stable and $\lambda \in \mathbb{C}^*$, then $(V, \lambda\Phi)$ is stable.
(3) Stability is an open condition in the sense that if (V, Φ) is stable then a dense open set of any holomorphic family of Higgs bundles containing (V, Φ) is also stable.

The most important property of stable Higgs bundles is given by the theorem of Simpson [13] and the author [6]: if (V, Φ) is stable and $\deg V = 0$ then there is a unique unitary connection A on V, compatible with the holomorphic structure, such that

$$F_A + [\Phi, \Phi^*] = 0 \in \Omega^{1,1}(\Sigma; \operatorname{End} V) \tag{2.1}$$

where F_A is the curvature of the connection. When $\Phi = 0$, this theorem becomes the well-known result of Narasimhan and Seshadri [11].

Conversely, if a pair (V, Φ) satisfies the Higgs bundle equations (2.1) then a vanishing theorem (see [6]) asserts that V is a direct sum of stable Higgs bundles. The same vanishing theorem also yields the result that any holomorphic section of V annihilated by Φ is necessarily covariant constant with respect to the connection A. This result has a number of implications. Note that (2.1) is a statement about a connection on a principal $U(n)$-bundle: the equation makes use only of the Lie bracket. Hence a holomorphic section of a vector bundle associated to any representation of $U(n)$ and annihilated by the action of Φ is covariant constant.

Similarly, (2.1) makes sense for a connection on a principal G-bundle P where G is the compact real form of a complex Lie group G^c and $A \mapsto -A^*$ is the compact real structure. The Higgs field Φ is now a holomorphic section of the bundle $\operatorname{ad} P \otimes_{\mathbb{C}} K$ where $\operatorname{ad} P$ is the Lie algebra bundle associated to the adjoint representation of G. A solution of the Higgs bundle equations therefore decomposes the holomorphic vector bundle $\operatorname{ad} P \otimes \mathbb{C}$ into a direct sum of stable Higgs bundles. Conversely, if the vector bundle $V = \operatorname{ad} P \otimes \mathbb{C}$ together with the Lie bracket action of Φ is a direct sum of Higgs bundles, we can solve the Higgs bundle equations and obtain a $U(\dim G)$-connection. Furthermore, the Lie bracket operation on V, considered as a section of $\operatorname{Hom}(V \otimes V, V)$, is holomorphic and compatible with Φ and hence covariant constant. The connection thus reduces to the group G.

If we take a connection A which solves the Higgs bundle equations (2.1), then

$$\nabla = \nabla_A + \Phi + \Phi^* \tag{2.2}$$

is a *flat* $GL(n, \mathbb{C})$-connection. Moreover, a vanishing theorem asserts that the holonomy action of $\pi_1(\Sigma)$ on $\mathbb{C}^n$ is completely reducible in the sense that any invariant subspace has an invariant complement. A theorem of Corlette [2] and Donaldson [3] gives the converse: if V is a vector bundle over a Riemann surface with a completely reducible flat connection ∇, then there exists a metric on V such that ∇ can be written in the form (2.2) where (A, Φ) satisfy the Higgs bundle equations (2.1). We may similarly extend this result to connections on principal G^c-bundles.

The two theorems of Corlette and Simpson provide the means for producing our results. We wish to describe a family of homomorphisms from $\pi_1(\Sigma)$ to a group $G^r \subset G^c$, or in other words, flat connections over a surface Σ. We do this by choosing a conformal structure on Σ and describing a certain family of stable Higgs bundles which, by appealing to Simpson's result, will yield a family of flat connections. Corlette's theorem will tell us that this family exhausts a component of the appropriate space of flat connections. To do this, we also need to understand some features of the moduli space concerned, details of which can again be found in the above references.

Standard arguments using the Atiyah–Singer index theorem and Banach space implicit function theorems show that the space of solutions to the Higgs bundle equations (2.1) (together with the holomorphicity condition $d''_A\Phi = 0$), modulo the group of gauge transformations, is a Hausdorff space $\mathbf{M}$ with a certain differentiable structure admitting singularities. We are here going to consider the moduli space from the principal bundle point of view, using the adjoint group (i.e. *G/centre* for a Lie group G). This introduces a few more singularities than the vector bundle situation, but is the natural Lie-theoretic context in which we choose to work. Here (see, for example [6] §5) the smooth points of the moduli space occur for pairs (A, Φ) for which there are no holomorphic sections of $\operatorname{ad} P \otimes \mathbb{C}$ annihilated by Φ. This is the generic, stable, situation: if we have a family of Lie algebra bundles and Higgs fields Φ containing a solution to the Higgs bundle equations at a smooth point of the moduli space, then the implicit function theorem tells us that there is a neighbourhood of that point for which corresponding solutions exist and these also give smooth points of the moduli space.

Viewed through the Higgs bundle interpretation, $\mathbf{M}$ is globally a normal quasi-projective variety [13], [12]. It has a holomorphic action of $\mathbb{C}^*$ corresponding to the scaling action on the Higgs field: $\Phi \mapsto \lambda\Phi$. Moreover, if the group G is simple and $p_1, \ldots, p_l$ form a basis for the algebra of invariant polynomials on the Lie algebra, of degrees $n_1, \ldots, n_l$ then we obtain a holomorphic map [7]

$$p\colon \mathbf{M} \to \bigoplus_{i=1}^{l} H^0(\Sigma; K^{n_i}) \tag{2.3}$$

defined by

$$p(A, \Phi) = (p_1(\Phi), \ldots, p_l(\Phi))$$

where $H^0(\Sigma; K^{n_i})$ is the vector space of holomorphic sections of K^{n_i}—the space of differentials of degree n_i on Σ. This map is proper.

In the light of Corlette's theorem, $\mathbf{M}$ is the moduli space of flat, completely reducible G^c-connections (where complete reducibility refers to the adjoint representation) and is thus identified with the space of completely reducible homomorphisms of the fundamental group to the complex adjoint group G^c, modulo the action of G^c. It is within this space that we shall identify components for the split real form G^r. We begin with the linear group case of $PSL(n, \mathbb{R})$.

§3. THE LINEAR CASE

The simplest of all Higgs bundles is the basic model for the ones we shall construct in this paper. Let $K^{1/2}$ be a square root of the canonical bundle (a "theta characteristic" or "spin structure" in other parlance) and define

$$V = K^{-1/2} \oplus K^{1/2}$$

with

$$\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Here 1 is to be interpreted as the canonical section of $\mathrm{Hom}(K^{1/2}; K^{-1/2} \otimes K)$.

The only Φ-invariant subbundle of V is $K^{-1/2}$ which has degree $(1 - g)$ and so for $g > 1$ this is a stable Higgs bundle.

The corresponding solution of the Higgs bundle equations consists of a hermitian metric on V which is the direct sum of a metric on $K^{-1/2}$ and its dual on $K^{1/2}$—in other words a Riemannian metric on Σ compatible with the complex structure. The equations (2.1) are then equivalent to the Gaussian curvature of this metric being equal to $-1/4$ (see [6]). Thus in this case uniformizing the surface is equivalent to solving the Higgs bundle equations.

We have here a Higgs bundle where both connection and Higgs field are defined on an $SU(2)$-principal bundle. Now take the n-fold symmetric product of $\mathbb{C}^2$ and the induced action of $SU(2)$—this is the action on homogeneous polynomials of degree n. We obtain an $(n + 1)$-dimensional *irreducible* representation of $SU(2)$. Consequently the vector bundle $W = S^n V$ with the induced action of Φ is a stable Higgs bundle. To be more explicit,

$$\begin{aligned} W &= S^n(K^{-1/2} \oplus K^{1/2}) \\ &= K^{-n/2} \oplus \cdots \oplus K^{n/2} \end{aligned}$$

and, with respect to this direct sum decomposition,

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & & & & 0 & 1 \\ 0 & 0 & \cdots & & 0 & 0 \end{pmatrix} \tag{3.1}$$

Consider now the Higgs bundle where we keep W as the holomorphic vector bundle and modify Φ to be a *companion matrix*:

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & & & & 0 & 1 \\ \alpha_n & \alpha_{n-1} & \cdots & \cdots & \alpha_1 & 0 \end{pmatrix} \tag{3.2}$$

where $\alpha_m \in H^0(\Sigma; K^{m+1})$.

By the openness of stability, we obtain a stable pair for sufficiently small α_m. However, consider the automorphism β of W defined by the matrix:

$$\beta = \begin{pmatrix} 1 & & & & \\ & \mu & & & \\ & & \mu^2 & & \\ & & & \ddots & \\ & & & & \mu^n \end{pmatrix}$$

Then

$$\beta^{-1}\Phi\beta = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \mu \\ \mu^{-n}\alpha_n & \mu^{1-n}\alpha_{n-1} & \cdots & \mu^{-1}\alpha_1 & 0 \end{pmatrix}$$

Now $(W, \beta^{-1}\Phi\beta))$ is stable from Property 1 in §2. From Property 2 so is $(W, \mu^{-1}\beta^{-1}\Phi\beta)$, thus the Higgs field defined by

$$\Psi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ \mu^{-n-1}\alpha_n & \mu^{-n}\alpha_{n-1} & \cdots & \mu^{-2}\alpha_1 & 0 \end{pmatrix}$$

is also stable. Hence, since μ occurs with negative powers as multiplier, if (3.2) is stable for sufficiently small α_m, it is in fact stable for all α_m by taking μ sufficiently large.

We now have a family of stable Higgs bundles parametrized by the vector space

$$\bigoplus_{m=1}^{n} H^0(\Sigma; K^{m+1}).$$

Recall now that a basis for the invariant polynomials on the Lie algebra of $PSL(n, \mathbb{C})$ is provided by the coefficients of the characteristic polynomial of a trace-free matrix:

$$\det(x - A) = x^{n+1} + p_1(A)x^{n-1} + \cdots + p_n(A).$$

For the companion matrix (3.1),

$$\det(x - \Phi) = x^{n+1} - \alpha_1 x^{n-1} - \alpha_2 x^{n-2} - \cdots - \alpha_n$$

and so our family of Higgs bundles provides a section s of the projection (2.3):

$$p: \mathbf{M} \to \bigoplus_{m=1}^{n} H^0(\Sigma; K^{m+1}).$$

One consequence of this is that the family is a *closed* subspace of the moduli space $\mathbf{M}$, for if a sequence $s(x_n)$ converges to $y \in \mathbf{M}$, then $x_n = ps(x_n)$ converges to $p(y)$. Hence $s(x_n)$ converges to $sp(y) = y$ in the same family.

To advance further we need to find an analogue of the symmetric power representation of $SU(2)$ in $SU(n+1)$ and of the companion matrix, for an arbitrary simple Lie group. These are provided by the theory of the principal 3-dimensional subgroup, which we review next.

§4. THE PRINCIPAL 3-DIMENSIONAL SUBGROUP

Details of the theory summarized here will be found in Kostant's three papers [8], [9] and [10].

Let $\mathfrak{g}^c$ be a complex simple Lie algebra of rank l. A nilpotent element $e \in \mathfrak{g}^c$ is called *regular* (or *principal*) if its centralizer is l-dimensional. Thus, for examle, in $\mathfrak{sl}(n, \mathbb{C})$ a regular nilpotent is conjugate to a matrix like (3.1), with one Jordan block.

By the Jacobson–Morozov lemma, any nilpotent element can be embedded in a 3-dimensional simple subalgebra (a copy of $\mathfrak{sl}(2, \mathbb{C})$) in $\mathfrak{g}^c$. It is generated by a semi-simple element x and nilpotent elements e and $\tilde{e}$ satisfying the relations:

$$[x, e] = e; \qquad [x, \tilde{e}] = -\tilde{e}; \qquad [e, \tilde{e}] = x.$$

For a regular nilpotent, this is called a *principal* 3-dimensional subalgebra.

Under the adjoint action of this subalgebra, the Lie algebra $\mathfrak{g}^c$ breaks up as a direct sum of irreducible representation spaces:

$$\mathfrak{g}^c = \bigoplus_{i=1}^{l} V_i. \tag{4.1}$$

Note that there are l summands—indeed, the highest weight vector e_i of each V_i under the action of x is annihilated by e, so the l-dimensional centralizer of e is spanned by $e_1, \ldots, e_l$. We take V_1 to be the 3-dimensional subalgebra itself, so $e = e_1$.

A principal 3-dimensional subalgebra defines a homomorphism from $SL(2, \mathbb{C})$ to G^c and hence from $SU(2)$ to a compact real form G of G^c. This is the generalization of the n-fold symmetric power of $\mathbb{C}^2$ which we considered above. We may take then the 3-dimensional subalgebra to be real with respect to the compact real form of G^c. If ρ is the anti-involution on $\mathfrak{g}^c$ defining the compact real form, then $\rho(x) = -x$ and $\rho(e) = \tilde{e}$ in the 3-dimensional subalgebra.

Each irreducible summand V_i is (see [8]) odd-dimensional and hence a representation space for $SO(3) = SU(2)/\pm 1$. In particular it is real. If its dimension is $(2m_i + 1)$ then the eigenvalues of $\operatorname{ad} x$, the semisimple element of $\mathfrak{sl}(2, \mathbb{C})$, on V_i are the integers m such that $-m_i \leq m \leq m_i$. The real Lie algebra $\mathfrak{g}$ of the compact real form breaks up as in (4.1) into the direct sum of these real representations.

Denote by $\mathfrak{g}_m$ the subspace of $\mathfrak{g}^c$ on which $\operatorname{ad} x$ acts with eigenvalue m. Then

$$\mathfrak{g}^c = \bigoplus_{m=-M}^{M} \mathfrak{g}_m \tag{4.2}$$

where $M = \max m_i$, and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

We now need to obtain the analogue of companion matrices, details of which can be found in [9].

We already encountered the highest weight vectors $e_1, \ldots, e_l$ of $V_1, \ldots, V_l$. Consider now elements of the form

$$f = \tilde{e}_1 + \alpha_1 e_1 + \cdots + \alpha_l e_l. \tag{4.3}$$

Theorem 7 of [9] proves that there exists a basis $p_1, \ldots, p_l$ of invariant polynomials on $\mathfrak{g}^c$ such that

$$p_i(f) = \alpha_i. \tag{4.4}$$

Moreover, the degree of p_i is $m_i + 1$, where $2m_i + 1$ is the dimension of V_i or equivalently m_i is the eigenvalue of $\operatorname{ad} x$ on e_i. These fundamental invariants of the Lie algebra are called the exponents.

Note that for the linear group $SL(n, \mathbb{C})$, the canonical form (4.3) is not quite the companion matrix (3.2). Instead it is a matrix of the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & & 0 \\ \alpha_1 & 0 & 1 & \cdots & & 0 \\ \alpha_2 & \alpha_1 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ \alpha_{n-1} & & & & \ddots & 1 \\ \alpha_n & \alpha_{n-1} & \cdots & \alpha_2 & \alpha_1 & 0 \end{pmatrix}$$

which is nevertheless conjugate to the companion matrix.

Finally, to put one's finger on regular nilpotents is easy. On the one hand, they form a conjugacy class which is open and dense in the set of all nilpotent elements. On the other, we can explicitly write down a regular nilpotent as follows: take a root system Δ with root vectors x_α, $(\alpha \in \Delta)$. Then

$$e = \sum_{\alpha \in \Delta^+} c_\alpha x_\alpha \tag{4.5}$$

is of course always nilpotent. It is *regular* iff $c_\alpha \neq 0$ for the simple roots α [8].

§5. THE GENERAL CASE

Given the algebra above, we shall now extend the construction of §3 to the case of a general simple Lie group.

We begin with the same basic $SU(2)$-Higgs bundle obtained from a metric of constant curvature and take the associated G-Higgs bundle defined by the compact principal 3-dimensional subgroup $SU(2) \to G$, G being the adjoint group of the compact real form of $\mathfrak{g}^c$. Recall that the connection of the basic solution reduces to $U(1)$, so the structure of the adjoint vector bundle $\operatorname{ad} P \otimes \mathbb{C}$ is determined by the action of $U(1)$ on $\mathfrak{g}^c$ or equivalently the eigenspaces of the semisimple element $\operatorname{ad} x$. From (4.2) we obtain the holomorphic structure

$$\operatorname{ad} P \otimes \mathbb{C} = \bigoplus_{m=-M}^{M} \mathfrak{g}_m \otimes K^m. \tag{5.1}$$

The Higgs field can then be written

$$\Phi = \tilde{e}_1 \tag{5.2}$$

Note that $[x, \tilde{e}_1] = -\tilde{e}_1$, so $\tilde{e}_1 \in \mathfrak{g}_{-1}$. Hence $\tilde{e}_1$ is to be thought of as a section of

$$(\mathfrak{g}_{-1} \otimes K^{-1}) \otimes K \subset \operatorname{ad} P \otimes_{\mathbb{C}} K.$$

We now define a deformation of this Higgs bundle by taking the same underlying principal G^c-bundle and modifying the Higgs field to

$$\Phi = \tilde{e}_1 + \alpha_1 e_1 + \cdots + \alpha_l e_l \tag{5.3}$$

where $\alpha_i \in H^0(\Sigma; K^{m_i+1})$. Again, since $e_i \in \mathfrak{g}_{m_i}$, $\alpha_i e_i$ is well-defined as a section of $\operatorname{ad} P \otimes_{\mathbb{C}} K$.

As remarked in §2, for sufficiently small α_i there will be a solution to the Higgs bundle equations, giving smooth points of the moduli space so long as there are no holomorphic (and hence covariant constant) sections of $\operatorname{ad} P$ commuting with Φ. Since the trivial representation does not occur in the V_i's (G^c is simple and so has no invariant linear polynomial), the basic solution has this property. Thus for small α_i, the vector bundle $\operatorname{ad} P$ and Higgs field Φ admit a solution to the Higgs bundle equations. It is therefore a direct sum of stable Higgs bundles as a vector bundle, a condition which is unchanged upon multiplying Φ by $\lambda \in \mathbb{C}^*$. Now consider the automorphism of $\operatorname{ad} P \otimes \mathbb{C}$ obtained by exponentiating x. This gives a $\mathbb{C}^*$-action which takes Φ in (5.3) to

$$\Psi = \mu^{-1}\tilde{e}_1 + \alpha_1 \mu^{m_1} e_1 + \cdots + \alpha_l \mu^{m_l} e_l.$$

Since all exponents m_i are positive, any Higgs field of the form (5.3) is equivalent to one of the form $\lambda\Phi$ where α_i is small. All members of the family are therefore sums of stable Higgs bundles and give solutions to the Higgs bundle equations.

The relation (4.4) shows that, as in the linear case, we obtain a section s of

$$p\colon \mathbf{M} \to \bigoplus_{i=1}^{l} H^0(\Sigma; K^{m_i+1})$$

and hence a closed subspace of the moduli space $\mathbf{M}$.

We therefore have a distinguished family of Higgs bundles, isomorphic to a vector space. From Corlette's theorem we can reinterpret this as a subspace of the moduli space of flat G^c-connections on Σ. What remains is to prove that the holonomy lies in the split real form G^r.

§6. REALITY

To determine the reality property of the flat connection we need more of the algebra of principal 3-dimensional subalgebras, contained in the following proposition:

PROPOSITION (6.1). *Let $\mathfrak{g}^c$ be a simple Lie algebra with principal 3-dimensional subalgebra $\langle x, e_1, \tilde{e}_1\rangle$ and let $e_1, \ldots, e_l$ be the highest weight vectors of the irreducible representations $V_1, \ldots, V_l \subset \mathfrak{g}^c$. Define σ by $\sigma(e_i) = -e_i$ and $\sigma(\tilde{e}_1) = -\tilde{e}_1$. Then,*

(1) *σ extends uniquely to a Lie algebra involution of $\mathfrak{g}^c$.*
(2) *The fixed point set of σ consists of the complexification of a maximal compact subalgebra of the split real form $\mathfrak{g}^r$ of $\mathfrak{g}^c$*

Proof. First note that *if* σ extends to an automorphism of $\mathfrak{g}^c$, then

$$\sigma((\operatorname{ad}\tilde{e}_1)^k e_i) = (-1)^{k+1}(\operatorname{ad}\tilde{e}_1)^k e_i. \tag{6.2}$$

Since the vectors $(\operatorname{ad}\tilde{e}_1)^k e_i$, $(0 \le k \le 2m_i + 1)$, form a basis for V_i, then σ is *determined* by the property $\sigma(e_i) = -e_i$, $\sigma(\tilde{e}_1) = -\tilde{e}_1$.

Now *define* the involution σ on $\mathfrak{g}^c = \bigoplus_{i=1}^{l} V_i$ by (6.2). We need to prove that it is a Lie algebra automorphism.

Let $w(a)$ denote the eigenvalue of the semisimple element x on a basis element a of the form $(\operatorname{ad}\tilde{e}_1)^k e_i$. Then (6.2) can be rewritten

$$\sigma(a) = (-1)^{m_i + w(a) + 1} a. \tag{6.3}$$

Thus, for basis vectors $a \in V_i$, $b \in V_j$

$$[\sigma(a), \sigma(b)] = (-1)^{m_i + m_j + w(a) + w(b)}[a, b]. \tag{6.4}$$

On the other hand,

$$[a, b] = \sum_{k=1}^{l} c_k$$

where each $c_k \in V_k$ has eigenvalue $w(a) + w(b)$ and (since each eigenspace of x in V_k is one-dimensional) is therefore a multiple of a basis vector.

Hence,

$$\sigma[a, b] = \sum_{k=1}^{l} (-1)^{w(a)+w(b)+1+m_k} c_k. \tag{6.5}$$

Thus to prove that $\sigma[a, b] = [\sigma(a), \sigma(b)]$, it suffices to prove that

$$m_i + m_j = m_k + 1 \qquad \mathrm{mod}\, 2. \tag{6.6}$$

Now the Lie bracket in $\mathfrak{g}^c$ defines an $SO(3)$-invariant element of

$$\mathrm{Hom}(V_i \otimes V_j, \oplus_{k=1}^{l} V_k)$$

Each V_i is isomorphic to the irreducible representation S^{2m_i}, the $2m_i$-fold symmetric power of the defining representation of $SU(2)$, and we have the Clebsch–Gordon decomposition of the tensor product:

$$S^{2m_i} \otimes S^{2m_j} \cong \oplus_{k=0}^{\min(m_i, m_j)} S^{2m_i+2m_j-2k}.$$

The projection onto the irreducible factor $S^{2m_i+2m_j-2k}$ is defined by contracting k times with the skew form ω on $\mathbb{C}^2$ which $SU(2)$ leaves invariant. The Lie bracket must factor through this decomposition.

Consider vectors $a_i \in V_i \cong S^{2m_i}$ and $a_j \in V_j \cong S^{2m_j}$ which are annihilated by $\mathrm{ad}\, x$. If we represent S^{2m} as homogeneous polynomials of degree $2m$ in the variables (z_1, z_2) then a_i is a multiple of $z_1^{m_i} z_2^{m_i}$. It is elementary to note that contracting $z_1^{m_i} z_2^{m_i}$ with $z_1^{m_j} z_2^{m_j}$ an odd number of times with ω gives zero (from the skew-symmetry of ω) and an even number $2k$ of times gives a non-zero multiple of $(z_1 z_2)^{m_i+m_j-2k}$. Now in the decomposition $\mathfrak{g}^c = \bigoplus_{i=1}^{l} V_i$, the vectors annihilated by $\mathrm{ad}\, x$ (the centralizer of x) constitute an l-dimensional subspace with one basis vector in each V_i. Since x is semi-simple this means that it is a regular element and hence its centralizer is abelian—a Cartan subalgebra. From the above discussion this means that the Lie bracket of a_i and a_j is zero, so that the corresponding element of $\mathrm{Hom}(V_i \otimes V_j, \oplus_{k=1}^{l} V_k)$ takes values only in those V_k for which

$$m_k = m_i + m_j - 2k - 1.$$

In other words, c_k in (6.5) is zero unless $m_i + m_j - m_k$ is odd. This is precisely statement (6.6), hence σ is indeed an automorphism of $\mathfrak{g}^c$.

Note in passing that the above argument shows that the highest weight representation $S^{2m_i+2m_j}$ does not occur in the image of the Lie bracket, and so $[e_i, e_j] = 0$ where e_i, e_j are highest weight vectors in V_i, V_j. Thus the centralizer of a regular nilpotent e_1 is abelian.

For the second part of the proposition, recall that a real form of a complex Lie algebra is an antilinear involution. If there is a Cartan subalgebra invariant by the involution on which the Killing form is *negative* definite, then we have a *compact* real form. If there is one for which it is *positive* definite, it is called a *split* (or normal) real form. Compact and split real forms are unique up to conjugation.

To link the automorphism σ defined above with split real forms, we need to find an appropriate Cartan subalgebra. We repeat here for convenience an argument of Kostant [8, Lemma 6.4A] giving the required construction.

Let x_α, $(\alpha \in \Delta)$, be root vectors which satisfy the standard relation

$$[x_\alpha, x_{-\alpha}] = \alpha \tag{6.7}$$

where the Killing form identifies roots with elements of a Cartan subalgebra $\mathfrak{h}$. The usual compact real form ρ is the antilinear extension of

$$\rho(x_\alpha) = x_{-\alpha}; \qquad \rho(\alpha) = -\alpha.$$

Let ψ be the highest root and in terms of the simple roots $\alpha_1, \ldots, \alpha_l$ write

$$\psi = \sum_{i=1}^{l} q_i \alpha_i \tag{6.8}$$

where $q_i > 0$. Now define

$$e_1 = \sum_{i=1}^{l} q_i^{1/2} x_{\alpha_i}$$

From (4.5), e_1 is a regular nilpotent element. Now if

$$\tilde{e}_1 = \sum_{i=1}^{l} q_i^{1/2} x_{-\alpha_i}$$

then

$$\begin{aligned}[][e_1, \tilde{e}_1] &= \sum_{i=1}^{l} q_i [x_{\alpha_i}, x_{-\alpha_i}] \\ &= \sum_{i=1}^{l} q_i \alpha_i \\ &= \psi\end{aligned}$$

and $\langle e_1, \psi, \tilde{e}_1 \rangle$ is a principal 3-dimensional subgroup invariant under the compact involution ρ.

We consider now $z = e_1 + x_{-\psi}$ and $\rho(z) = \tilde{e}_1 + x_\psi$. Then,

$$\begin{aligned}[][z, \rho(z)] &= [e_1 + x_{-\psi}, \tilde{e}_1 + x_\psi] \\ &= \psi - \psi \\ &= 0\end{aligned}$$

since $[x_{\alpha_i}, x_\psi] = 0$ as ψ is the highest root.

Thus z is normal and hence semi-simple. The subalgebra we seek is its centralizer. To show this we consider y such that $[y, z] = 0$ and write

$$y = v + x + u$$

where $v \in \mathfrak{n}_-$, the nilpotent subalgebra generated by the negative root vectors, $x \in \mathfrak{h}$, the Cartan subalgebra, and $u \in \mathfrak{n}_+$, generated by positive root vectors. Then,

$$0 = [z, y] = [e_1 + x_{-\psi}, v + x + u] \tag{6.10}$$

Now $[x_{-\psi}, v] = 0$ since ψ is the highest root, and similarly $[x_{-\psi}, u] \in \mathfrak{h} \oplus \mathfrak{n}_-$. Since $x \in \mathfrak{h}$, then $[x_{-\psi}, x] = -\psi(x) x_{-\psi}$. Also, $[e_1, v] \in \mathfrak{h} \oplus \mathfrak{n}_-$. Thus, projecting (6.10) onto $\mathfrak{n}_+$ gives

$$[e_1, u] + [e_1, x] = 0.$$

But $[e_1, x]$ lies in the span of $x_{\alpha_1}, \ldots, x_{\alpha_l}$ and $[e_1, u]$ involves higher root spaces, so each term vanishes separately:

$$[e_1, u] = 0; \qquad [e_1, x] = 0$$

Now the centralizer of e_1 consists of nilpotent elements so, x being semi-simple, we must have $x = 0$. Thus $y = v + u$ where u commutes with e_1.

Suppose now that $u = 0$, then (6.10) gives

$$0 = [e_1 + x_{-\psi}, v] = [e_1, v]$$

But the centralizer of e_1 lies in $\mathfrak{n}_+$ so $v = 0$ and y itself vanishes. The map $y \mapsto u$ therefore gives an injective linear map from the centralizer of z to the centralizer of e_1, which is l-dimensional from the regularity of e_1. Consequently, z has centralizer of dimension $\leq l$. the rank of $\mathfrak{g}^c$, and hence equal to l. Thus z is regular and its centralizer is a Cartan subalgebra $\mathfrak{h}'$. Moreover, since $[z, \rho(z)] = 0$ it is also the centralizer of $\rho(z)$ and is thus preserved by the compact real structure ρ.

The principal 3-dimensional subgroup is real with respect to ρ, as are the irreducible subspaces V_i. The involution σ, alternately ± 1 on the weight spaces of V_i, preserves the real structure, so $\sigma\rho = \rho\sigma = \tau$ is an antilinear involution of $\mathfrak{g}^c$ giving another real structure. Now since by definition σ acts as -1 on the centralizer of e_1 and $\tilde{e}_1$, it acts as -1 on the Cartan subalgebra $\mathfrak{h}'$, each of whose elements is of the form $y = u + v$, with u, v in the centralizers of $e_1, \tilde{e}_1$. Thus τ preserves $\mathfrak{h}'$ and moreover if $y \in \mathfrak{h}'$ is fixed by ρ, then

$$\tau(iy) = \sigma\rho(iy) = \sigma(-iy) = iy$$

Hence since the Killing form is negative definite on the fixed point set of ρ, it is positive definite on the fixed point set of τ. Thus τ defines a *split* real form $\mathfrak{g}^r$ of $\mathfrak{g}^c$.

Finally, decompose $\mathfrak{g}^c$ as a *real* vector space into a direct sum of common eigenspaces of the commuting involutions σ and ρ:

$$\mathfrak{g}^c = \mathfrak{g}^+_+ \oplus \mathfrak{g}^+_- \oplus \mathfrak{g}^-_+ \oplus \mathfrak{g}^-_-$$

where the upper index gives the sign of the σ-eigenvalue and the lower one the ρ-eigenvalue. The fixed point set of σ is

$$\mathfrak{g}^\sigma = \mathfrak{g}^+_+ \oplus \mathfrak{g}^+_-.$$

This is a complex Lie algebra with real structure defined by ρ—the complexification of the real Lie algebra $\mathfrak{g}^+_+$ on which the Killing form is negative and hence $\mathfrak{g}^+_+$ is a compact real form of $\mathfrak{g}^\sigma$.

On the other hand, the split real form $\mathfrak{g}^r$ is the fixed point set of $\sigma\rho$, so

$$\mathfrak{g}^r = \mathfrak{g}^+_+ \oplus \mathfrak{g}^-_-.$$

Here the Killing form is negative on $\mathfrak{g}^+_+$ and positive on $\mathfrak{g}^-_-$, so $\mathfrak{g}^\sigma$ is the complexification of the maximal compact subalgebra $\mathfrak{g}^+_+$ of $\mathfrak{g}^r$ as required.

Remarks (6.11).

1. The involution σ of Proposition (6.1) has been defined here in a uniform way for all simple Lie algebras. In fact P. Slodowy has pointed out that for Lie algebras of type B_n, C_n, D_{2n}, E_7, E_8, F_4 and G_2 it is the inner automorphism defined by a rotation of π in the principal 3-dimensional subgroup $SO(3)$. For the remaining cases of A_n, D_{2n+1} and E_6 this involution composed with the outer automorphism corresponding to the 2-fold symmetry of the Dynkin diagram gives σ.

2. Since both σ and ρ in the proof of the Proposition preserve the principal 3-dimensional subgroup, it follows that there is a split principal 3-dimensional subgroup, a homomorphism from $PSL(2, \mathbb{R})$ to the split adjoint group G^r. In the linear case this is clear since the defining representation of $SL(2, \mathbb{R})$ is real and so therefore is its nth symmetric power, giving a homomorphism from $SL(2, \mathbb{R})$ to $SL(n + 1, \mathbb{R})$.

§7. FLAT CONNECTIONS

As pointed out in §2, a solution of the Higgs bundle equations for a compact group G defines a flat G^c-connection

$$\nabla = \nabla_A + \Phi + \Phi^*. \tag{7.1}$$

We here define $x \mapsto -x^*$ as the compact real structure ρ on $\mathfrak{g}^c$ in the general case.

For the family of Higgs bundles constructed in §5, we have a Higgs field defined by (5.3):

$$\Phi = \tilde{e}_1 + \alpha_1 e_1 + \cdots + \alpha_l e_l$$

and from §6 there is a natural involution on the Lie algebra bundle (5.1):

$$\operatorname{ad} P \otimes \mathbb{C} = \bigoplus_{m=-M}^{M} \mathfrak{g}_m \otimes K^m$$

for which $\sigma(\Phi) = -\Phi$. Let (A, Φ) be the corresponding solution to the Higgs bundle equations (2.1):

$$F_A + [\Phi, \Phi^*] = 0.$$

Then clearly $(A, -\Phi)$ is also a solution. But so is $(\sigma^* A, \sigma^* \Phi) = (\sigma^* A, -\Phi)$. From the uniqueness part of Simpson's theorem (a consequence of the basic principle that holomorphic Φ-invariant objects are covariantly constant) we see that

$$\sigma^* A = A. \tag{7.2}$$

In other words, the holonomy of the connection A reduces to the intersection of G and the fixed point set of σ, which from Proposition (6.1) is contained in the maximal compact subgroup K of the split real form G^r of G^c.

As we saw in §6, the antilinear involution $\tau = \rho\sigma$ defines the split real form, so in our context we have a reduction of the principal bundle from G^c to G^r. Moreover,

$$\begin{aligned} \tau^* A &= (\rho\sigma)^* A \\ &= \sigma^* A \qquad \text{since } A \text{ is a } G\text{-connection} \\ &= A \qquad \text{from (7.2)} \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} \tau^*(\Phi + \Phi^*) &= (\rho\sigma)^*(\Phi - \rho^*(\Phi)) \\ &= \sigma^*(\rho^*(\Phi) - \Phi) \\ &= -\rho^*(\Phi) + \Phi \qquad \text{since } \sigma^*\Phi = -\Phi \\ &= \Phi + \Phi^* \end{aligned}$$

Thus the connection $\nabla_A + \Phi + \Phi^*$ has holonomy contained in G^r.

From this we directly obtain our main theorem:

THEOREM (7.5). *Let G^c be the adjoint group of a complex simple Lie Group and G^r its split real form. Let $m_1, \ldots, m_l$ be the exponents of G^c and let $\alpha_1, \ldots, \alpha_l$ be holomorphic differentials of degree $m_i + 1$ on a compact Riemann surface Σ of genus $g > 1$. Then the solution to the Higgs bundle equations corresponding to*

$$\operatorname{ad} P \otimes \mathbb{C} = \bigoplus_{m=-M}^{M} \mathfrak{g}_m \otimes K^m$$

$$\Phi = \tilde{e}_1 + \sum_{i=1}^{l} \alpha_i e_i$$

defines an isomorphism from the vector space

$$\bigoplus_{i=1}^{l} H^0(\Sigma; K^{m_i+1})$$

to a component of the moduli space of flat completely reducible G^r-connections on Σ.

Proof. We have already seen in §5 that these solutions provide a section s of $p: \mathbf{M} \to \bigoplus_{i=1}^{l} H^0(\Sigma; K^{m_i+1})$ taking values in the smooth points of $\mathbf{M}$. Corlette's theorem identifies $\mathbf{M}$ as the moduli space of completely reducible flat G^c-connections and the argument above shows that for our family these are actually G^r-connections. The image of s is therefore a closed submanifold of this space of connections (the derivative of s is injective since $p \circ s$ is the identity).

But now the dimension of the moduli space of flat G^r-connections is (by using the index theorem, for example)

$$2(g-1)\dim G^r.$$

On the other hand, the real dimension of the vector space $\bigoplus_{i=1}^{l} H^0(\Sigma; K^{m_i+1})$ is, by the Riemann–Roch theorem, equal to:

$$\begin{aligned} 2\sum_{i=1}^{l}(2m_i+1)(g-1) &= 2(g-1)\sum_{i=1}^{l}\dim_{\mathbb{C}} V_i &&\text{(cf. §4)}\\ &= 2(g-1)\dim_{\mathbb{C}}\mathfrak{g}^c &&\text{from (4.1)}\\ &= 2(g-1)\dim G^r \end{aligned}$$

Thus the image of s is an open and closed submanifold which is connected and is hence a *component.*

We shall call this component the *Teichmüller component*. In the case of $G^r = PSL(2, \mathbb{R})$ it can be directly identified with Teichmüller space, the space of equivalence classes of conformal structures on the surface Σ, modulo diffeomorphisms homotopic to the identity. A proof of this is given in [6], where the quadratic differential $\alpha_1 \in H^0(\Sigma; K^2)$ in the theorem is used to define a metric of constant negative curvature. If, in the general family of (7.5), we take $\alpha_2 = \alpha_3 = \cdots = \alpha_l = 0$, then we have an embedding of Teichmüller space in the Teichmüller component. This is nothing more than the space of flat G^r-connections associated to the uniformizing representations of $\pi_1(\Sigma)$ in $PSL(2, \mathbb{R})$ by the split principal 3-dimensional subgroup $PSL(2, \mathbb{R}) \subset G^r$ of Remark (6.11). These generalized Teichmüller spaces are therefore extensions of ordinary Teichmüller space.

§8. OTHER COMPONENTS

To understand better the rôle of the Teichmüller component we shall here calculate all the components of the space of flat completely reducible connections in the linear case—the split real form $PSL(n, \mathbb{R})$. Note that in general we can use the *topological* classification of principal G^r-bundles to provide at least some separation into disjoint classes of spaces of connections. This is equivalent to classifying principal K-bundles where K is the maximal compact subgroup of G^r. In the case where $G^r = Sp(2n, \mathbb{R})$, then $K = U(n)$, so we have in particular an integer invariant—the Chern class. The linear case is somewhat easier, however, since the maximal compact subgroup in the adjoint group $PSL(n, \mathbb{R})$ is $SO(n)$ for n odd and $SO(n)/\pm 1$ for n even. The topological classification here for $n > 2$ is given by elements of the cohomology group $H^2(\Sigma; Z)$ where Z is the centre of $Spin(n)$, the simply-connected covering group. We obtain:

$$\begin{aligned} H^2(\Sigma; Z) &\cong \mathbb{Z}_2 && n \text{ odd} \\ &\cong \mathbb{Z}_4 && n = 2 \bmod 4 \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 && n = 0 \bmod 4. \end{aligned} \tag{8.1}$$

Our strategy for finding the components within a given topological class will be to solve the Higgs bundle equations and consider the function f defined by:

$$f(A, \Phi) = i \int_\Sigma \operatorname{tr}(\Phi\Phi^*). \tag{8.2}$$

It is a consequence of Uhlenbeck's weak compactness theorem (see [6]) that this non-negative gauge-invariant function is *proper* on the moduli space $\mathbf{M}$ of Higgs bundles. Thus on each component of the closed subspace of $\mathbf{M}$ which gives flat $PSL(n, \mathbb{R})$-connections it must have a minimum. Our primary aim will therefore be to seek all local minima of f on the corresponding space. For simplicity we shall first consider only those connections which lift to $SL(n, \mathbb{R})$, since in particular this is where the Teichmüller component lies.

Firstly, we should determine which Higgs bundles give flat $SL(n, \mathbb{R})$-connections. For this, note that the involution σ on $\mathfrak{sl}(n, \mathbb{C})$ which relates the compact real form ρ and the split real form τ can be taken to be

$$\sigma(x) = -x^T.$$

Now if the connection A has holonomy in $SO(n)$ and $\Phi = \Phi^T$, then $\sigma^*(A) = A$ and $\sigma^*\Phi = -\Phi$, so the arguments of (7.3) and (7.4) show that

$$\nabla_A + \Phi + \Phi^*$$

has holonomy in $SL(n, \mathbb{R})$. Conversely, the methods of Corlette and Donaldson for finding the Higgs bundle from a flat connection involve a reduction from $SL(n, \mathbb{R})$ to its maximal compact subgroup by means of a harmonic section of the associated $SL(n, \mathbb{R})/SO(n)$ bundle, which forces A and Φ to be as above.

In purely holomorphic terms, we seek a Higgs bundle consisting of a vector bundle V with a non-degenerate *quadratic form* (an *orthogonal* bundle) and a Higgs field $\Phi \in H^0(\Sigma; \operatorname{End} V \otimes K)$ which is trace-free and symmetric with respect to that form. Thus we have a non-degenerate holomorphic quadratic form

$$Q\colon V \otimes V \to \mathbb{C} \tag{8.3}$$

and such that

$$Q(\Phi v, w) = Q(v, \Phi w) \in K. \tag{8.4}$$

The function $f = i\int_\Sigma \operatorname{tr}(\Phi\Phi^*)$ has a close relationship with the circle action

$$\Phi \mapsto e^{i\theta}\Phi.$$

In fact, if we consider the space $\mathcal{A} \times \Omega$ of pairs (A, Φ) (where A is a G-connection and $\Phi \in \Omega^{1,0}(\Sigma; \operatorname{ad} P \otimes \mathbb{C})$) as an infinite-dimensional flat Kähler manifold with Kähler metric

$$\|(\dot{A}, \dot{\Phi})\|^2 = \int \operatorname{tr}(\dot{A} \wedge *\dot{A} + \dot{\Phi}^* \wedge \dot{\Phi})$$

then f is the *moment map* of the circle action in the sense that

$$\operatorname{grad} f = IX \tag{8.5}$$

where X is the vector field generating the circle action. This property descends to the moduli space $\mathbf{M}$ which has (on its smooth points) a natural Kähler metric induced from the one above and compatible with its holomorphic structure as the moduli space of stable Higgs bundles (V, Φ), (see [6]). In particular, (8.5) shows that smooth critical points of f are fixed points of the circle action $\Phi \mapsto e^{i\theta}\Phi$. These are of two types: where $\Phi = 0$ or where there is a 1-parameter group of covariant constant gauge transformations which induces the action on Φ. The involution $\Phi \mapsto -\Phi^T$ commutes with the circle action, so the critical points of f on the subspace $\mathbf{M}_0$ of orthogonal bundles V and symmetric Higgs fields Φ (which is determined by the involution) are still critical points on $\mathbf{M}$.

Hence, finding local minima of f involves first finding the smooth fixed points of the circle action on $\mathbf{M}_0$ and identifying those for which the Hessian is non-negative. The singular points of $\mathbf{M}_0$ consist of direct sums of stable Higgs bundles of lower rank and so in particular sums of lower rank smooth minima. A second variation argument at these singular points must than be used to examine the reducible cases.

To analyse the second variation, note that in all cases the 1-parameter group of covariant constant gauge transformations is generated by $\psi \in \Omega^0(\Sigma; \operatorname{ad} P)$ such that

$$d_A\psi = 0 \tag{8.6}$$

and

$$[\psi, \Phi] = i\Phi; \qquad [\psi, \Phi^*] = -i\Phi^*. \tag{8.7}$$

To check that this gives a critical point of f, we note that

$$\begin{aligned} \dot{f} &= i\int_\Sigma \operatorname{tr}(\dot{\Phi}\Phi^* + \Phi\dot{\Phi}^*) \\ &= i\int_\Sigma \operatorname{tr}(i\dot{\Phi}[\psi, \Phi^*] - i[\psi, \Phi]\dot{\Phi}^*) \qquad \text{by (8.7)} \\ &= -i\int_\Sigma \operatorname{tr}(i\psi([\Phi, \dot{\Phi}^*] + [\dot{\Phi}, \Phi^*])) \\ &= i\int_\Sigma \operatorname{tr}(i\psi\, d_A\dot{A}). \end{aligned}$$

This last formula follows since $(\dot{A}, \dot{\Phi})$ satisfies the linearization of the Higgs bundle equations (2.1):

$$d_A\dot{A} + [\dot{\Phi}, \Phi^*] + [\Phi, \dot{\Phi}^*] = 0. \tag{8.8}$$

But now

$$\int_\Sigma \operatorname{tr}(\psi d_A\dot{A}) = -\int_\Sigma \operatorname{tr}(d_A\psi\dot{A}) = 0$$

from (8.6), so f is certainly critical at these points.

Taking a second derivative of (8.8) gives:

$$d_A\ddot{A} + [\ddot{\Phi}, \Phi^*] + [\Phi, \ddot{\Phi}^*] + [\dot{A}, \dot{A}] + 2[\dot{\Phi}, \dot{\Phi}^*] = 0 \tag{8.9}$$

and then

$$\ddot{f} = i\int_\Sigma \operatorname{tr}(\ddot{\Phi}\Phi^* + \Phi\ddot{\Phi}^*) + \operatorname{tr}(\dot{\Phi}\dot{\Phi}^* + \dot{\Phi}\dot{\Phi}^*).$$

Making a similar substitution as above, but with (8.9) replacing (8.8) gives

$$\begin{aligned} \ddot{f} &= i \int_{\Sigma} \operatorname{tr}(i\psi([\dot{A}, \dot{A}] + 2[\dot{\Phi}, \dot{\Phi}^*])) + 2 \operatorname{tr} \dot{\Phi}\dot{\Phi}^* \\ &= i \int_{\Sigma} \operatorname{tr}(i[\psi, \dot{A}]\dot{A} + 2i[\psi, \dot{\Phi}]\dot{\Phi}^* + 2\dot{\Phi}\dot{\Phi}^*). \end{aligned} \tag{8.10}$$

Now $\dot{A}$ represents a variation in the holomorphic structure of V and $\dot{\Phi}$ a variation in the Higgs field. The natural complex structure on both these spaces of variations is defined by

$$(\dot{A}, \dot{\Phi}) \mapsto (- * \dot{A}, i\dot{\Phi})$$

recalling that the (0, 1)-component of the connection defines the holomorphic structure.

Suppose the circle acts with weights m, n on the complex vector $(\dot{A}, \dot{\Phi})$, then

$$[\psi, \dot{A}] = - m * \dot{A}; \qquad [\psi, \dot{\Phi}] = in\dot{\Phi}$$

and substituting in (8.10) we obtain

$$\ddot{f} = i \int_{\Sigma} \operatorname{tr}(- im * \dot{A} \wedge \dot{A} - (2n - 2)\dot{\Phi}\dot{\Phi}^*).$$

Thus, if $m > 0$ or $n > 1$, the variation is negative in certain directions. Note that the argument here works for both smooth and singular points of the moduli space *provided* the infinitesimal variation $(\dot{A}, \dot{\Phi})$ is tangent to an actual smooth family of deformations.

§9. LOCAL MINIMA—THE HOLOMORPHIC VIEWPOINT

The space T of infinitesimal deformations of the Higgs bundle equations can be defined using elliptic complexes as in ([6] §5) but also has an equivalent description using sheaf cohomology groups as in [12]. There is an exact sequence (9.1):

$$0 \to H^0(\Sigma; \operatorname{ad} P \otimes \mathbb{C}) \xrightarrow{\alpha} H^0(\Sigma; \operatorname{ad} P \otimes K) \xrightarrow{\beta} \mathrm{T}$$

$$\xrightarrow{\gamma} H^1(\Sigma; \operatorname{ad} P \otimes \mathbb{C}) \xrightarrow{\delta} H^1(\Sigma; \operatorname{ad} P \otimes K) \to 0.$$

Here α and δ are defined by the Lie bracket with the Higgs field $\Phi \in H^0(\Sigma; \operatorname{ad} P \otimes K)$. The homomorphism γ is the natural map associating to an infinitesimal variation $(\dot{A}, \dot{\Phi})$ the class represented by $\dot{A}^{0,1} \in \Omega^{0,1}(\Sigma; \operatorname{ad} P \otimes \mathbb{C})$. For a smooth point of M, T is the tangent space, so that every vector in T is tangent to an actual deformation. At a singular point this is not necessarily so, and each vector must be individually analysed.

We are concerned here not with the whole tangent space to M, but with that of the subspace M_0. Here the bundle has a quadratic form (8.3), defining an isomorphism from V to V^*, and an infinitesimal deformation of this structure lies in the cohomology group of skew-symmetric endomorphisms $H^1(\Sigma; \Lambda^2 V)$. The Higgs field is symmetric and trace-free and therefore lies in the space $H^0(\Sigma; S_0^2 V \otimes K)$, the subscript 0 denoting trace-free. The infinitesimal deformation space $\mathrm{T}_0 \subset \mathrm{T}$ then fits into an exact sequence (9.2):

$$0 \to H^0(\Sigma; \Lambda^2 V) \xrightarrow{\alpha} H^0(\Sigma; S_0^2 V \otimes K) \xrightarrow{\beta} \mathrm{T}_0$$

$$\xrightarrow{\gamma} H^1(\Sigma; \Lambda^2 V) \xrightarrow{\delta} H^1(\Sigma; S_0^2 V \otimes K) \to 0.$$

Thus, at a local minimum of f at a smooth point of $\mathbf{M}_0$, the circle action must have weights $m_i \leq 0$ on $\operatorname{Ker}\delta$ and weights $n_j \leq 1$ on $\operatorname{Coker}\alpha$. This will be our criterion for identifying the irreducible local minima.

Let (V, Φ) be a stable Higgs bundle which represents a fixed point of the circle action. Then the eigenspaces of the infinitesimal gauge transformation ψ break V up into a direct sum of sub-bundles:

$$V = \bigoplus_m U_m \tag{9.3}$$

where ψ acts as im on U_m. Since

$$[\psi, \Phi] = i\Phi,$$

then

$$\Phi\colon U_m \to U_{m+1} \otimes K. \tag{9.4}$$

(This is called by Simpson [13] a "variation of Hodge structure".) In our case ψ is skew-symmetric so we have:

$$Q(\psi u_m, u_n) = -Q(u_m, \psi u_n)$$

and $Q(u_m, u_n) = 0$ unless $m + n = 0$. Since, moreover, Q is non-degenerate, we have

$$U_{-m} \cong U_m^*. \tag{9.5}$$

(Note that as the consecutive eigenvalues of ψ differ by 1, (9.5) implies that they are integers when there is an odd number of summands, and half-integers for an even number.) Finally, the Higgs field Φ is symmetric, so under the isomorphism (9.5) $\Phi\colon U_{-m} \to U_{-m+1} \otimes K$ transforms to the dual of $\Phi\colon U_{m-1} \to U_m \otimes K$.

Our first step towards finding the minima of f is the following (unfortunately rather long) lemma:

LEMMA (9.6). *Let (V, Φ) be a stable Higgs bundle in $\mathbf{M}_0$ which is a local minimum for f. Then either $\Phi = 0$ or each U_m in the decomposition* (9.3) *is a line bundle.*

Proof. From (9.5) we have $V = \bigoplus_{m=-n}^{n} U_m$. We consider the top component U_n first. Now from (9.4),

$$\Phi(U_n) = 0$$

and so U_n is Φ-invariant. By stability (cf. §2),

$$\deg U_n / \operatorname{rk} U_n < \deg V / \operatorname{rk} V \tag{9.7}$$

but since $V \cong V^*$, we have $\deg V = 0$. Hence,

$$\deg U_n < 0.$$

In general, let $d_m = \deg U_m$ and $r_m = \operatorname{rk} U_m$, so

$$d_n \leq -1. \tag{9.8}$$

Now consider $\Lambda^2 U_n \subset \Lambda^2 V$. We find

$$\deg \Lambda^2 U_n = (r_n - 1) d_n$$

which, from (9.8), is negative if $r_n > 1$.

Hence by the Riemann–Roch theorem,

$$\dim H^1(\Sigma; \Lambda^2 U_n) \geq -\deg \Lambda^2 U_n + (g-1) \operatorname{rk} \Lambda^2 U_n > 0.$$

Therefore, the space $H^1(\Sigma; \Lambda^2 U_n)$, which is a direct summand in $H^1(\Sigma; \Lambda^2 V)$ with positive weight, is non-trivial. On the other hand, recall that the map $\delta: H^1(\Sigma; \Lambda^2 V) \to H^1(\Sigma; S_0^2 V \otimes K)$ is given by the Lie bracket with Φ. Since $\Phi(U_n) = 0$, $H^1(\Sigma; \Lambda^2 U_n)$ is in $\operatorname{Ker} \delta$ and so this contradicts the assumption of minimality of f. It follows that we must have $r_n = 1$ and U_n is a line bundle. Ultimately, we shall proceed inductively from this point of departure, but we need to establish some estimates first.

Suppose U_m is a line bundle. Now $\Phi: U_{m-1} \to U_m K$ is not identically zero, since we are assuming that (V, Φ) is stable, and in particular irreducible, as a Higgs bundle. Thus Φ is generically surjective and there exists a sub-bundle $W_{m-1} \subset U_{m-1}$ such that $\Phi(W_{m-1}) = 0$ and $U_{m-1}/W_{m-1} = L_{m-1}$ is a line bundle. The homomorphism $\Phi: U_{m-1} \to U_m K$ then factors through a homomorphism of line bundles $\phi: L_{m-1} \to U_m K$. In particular, we must have

$$\deg L_{m-1} \leq \deg U_m K = d_m + (2g - 2).$$

Now since $\Phi(W_{m-1}) = 0$ and $\Phi(U_n) = 0$, the image of $H^1(\Sigma; W_{m-1} \otimes U_n)$ in the cohomology group $H^1(\Sigma; U_{m-1} \otimes U_n)$ is annihilated by Φ and has positive weight if $m - 1 + n > 0$. By the minimality of f it must be zero, and hence from the exact cohomology sequence of

$$0 \to W_{m-1} \otimes U_n \to U_{m-1} \otimes U_n \to L_{m-1} U_n \to 0$$

the coboundary map (or Bockstein)

$$B: H^0(\Sigma; L_{m-1} U_n) \to H^1(\Sigma; W_{m-1} \otimes U_n) \tag{9.9}$$

must be surjective. We shall estimate the dimensions of these spaces.

Firstly, from Riemann-Roch,

$$\begin{aligned} \dim H^1(\Sigma; W_{m-1} \otimes U_n) &\geq -\deg(W_{m-1} \otimes U_n) + (g-1)\operatorname{rk} W_{m-1} \\ &= -\deg W_{m-1} + (g - 1 - d_n)\operatorname{rk} W_{m-1}. \end{aligned} \tag{9.10}$$

Now since $\Phi(W_{m-1}) = 0$, W_{m-1} is a Φ-invariant sub-bundle of V and so by the stability of (V, Φ), if $W_{m-1} \neq 0$,

$$\deg W_{m-1} \leq -1.$$

Thus (9.10) gives

$$\dim H^1(\Sigma; W_{m-1} \otimes U_n) \geq g - d_n. \tag{9.11}$$

Now consider the dimension of $H^0(\Sigma; L_{m-1} U_n)$. From (9.11) and (9.8) if $\operatorname{rk} W_{m-1} > 0$ this is at least $g + 1$, since B in (9.9) is surjective. Hence the degree of $L_{m-1} U_n$ is at least $2g$. Since $\deg L_{m-1} \leq \deg U_m K = d_m + 2g - 2$, this means that

$$d_m + d_n \geq 2. \tag{9.12}$$

As a first example of the use of this, take $m = n$ and then (9.12) gives $d_n \geq 1$, contradicting (9.8). We deduce immediately that $\operatorname{rk} W_{n-1} = 0$ and therefore U_{n-1} is a line bundle. Taking $m = n - 1$ gives $d_{n-1} + d_n \geq 2$. But $U_{n-1} \oplus U_n$ is a Φ-invariant sub-bundle, so by stability $d_{n-1} + d_n \leq -1$. This is again a contradiction, so U_{n-2} must be a line bundle.

We introduce now another argument which will give more control over the degrees d_m. If U_m and U_{m+1} are line bundles, then since (V, Φ) is irreducible, $\Phi: U_m \to U_{m+1} K$ is a non-zero homomorphism. Take the induced map on cohomology:

$$\phi_1: H^1(\Sigma; U_m U_n) \to H^1(\Sigma; U_{m+1} U_n K).$$

This map is surjective since the quotient sheaf is supported on points. But as $\Phi(U_n) = 0$, the kernel of ϕ_1 is a subspace of $H^1(\Sigma; \Lambda^2 V)$ for $(m < n)$ which is annihilated by Φ and is of positive weight if $m + n > 0$. By minimality the kernel must be zero, so ϕ_1 is an isomorphism.

Now consider

$$S^2 U_{m+1} \subseteq S_0^2(\oplus U_l) = S_0^2 V.$$

If $2m + 2 > 1$, then any $s \in H^0(\Sigma; S^2 U_{m+1} \otimes K)$ must be in the image of $\Phi: \Lambda^2 V \to S_0^2 V \otimes K$ by the minimality of f, for otherwise the cokernel would have a subspace of weight > 1. In the decomposition

$$\Lambda^2 V = \Lambda^2(\oplus U_l)$$

we have $\Phi(\Lambda^2 U_l) \subseteq U_l \otimes U_{l+1} \otimes K$ and

$$\Phi(U_l \otimes U_k) \subseteq U_{l+1} \otimes U_k \otimes K + U_l \otimes U_{k+1} \otimes K.$$

Thus if $s = \Phi(u)$ is a section of $S^2 U_{m+1} \otimes K$, then u must have a non-zero component u_0 in $U_m \otimes U_{m+1}$. If P denotes the projection from $S_0^2 V \otimes K$ onto $S^2 U_{m+1} \otimes K$, then

$$s = P\Phi(u_0) = \phi_0(u_0)$$

where the map ϕ_0 from $U_m \otimes U_{m+1}$ to $S^2 U_{m+1} \otimes K$ is just the symmetrization of $\Phi \otimes 1$. We see therefore that

$$\phi_0: H^0(\Sigma; U_m \otimes U_{m+1}) \to H^0(\Sigma; S^2 U_{m+1} \otimes K)$$

is surjective if $2m + 1 > 0$. However, if as above U_m and U_{m+1} are line bundles, ϕ_0 is injective, for it is simply multiplication by a non-zero section of $U_m^* U_{m+1} K$. In this case, then, ϕ_0 is an isomorphism.

To summarize, if U_m and U_{m+1} are line bundles

$$\phi_1: H^1(\Sigma; U_m U_n) \to H^1(\Sigma; U_{m+1} U_n K) \tag{9.13}$$

and

$$\phi_0: H^0(\Sigma; U_m U_{m+1}) \to H^0(\Sigma; U_{m+1}^2 K) \tag{9.14}$$

are isomorphisms if $m + n > 0$ and $2m + 1 > 0$.

A special case of the above argument is to consider the isomorphisms ϕ_0 for $m = n$ and ϕ_1 for $m = n - 1$. These give isomorphisms of H^0 and H^1 for the line bundles $U_n U_{n-1}$ and $U_n^2 K$. In particular, the Riemann–Roch theorem implies that they have the same degree, so $\Phi \in U_{n-1}^* U_n K$ is a non-trivial section of a line bundle of degree zero and hence an isomorphism. Thus:

$$U_{n-1} \cong U_n K \tag{9.15}$$

and $d_{n-1} = d_n + 2(g - 1)$.

We have seen already that U_{n-2} is a line bundle, so we can consider (9.13) and (9.14) for $m = n - 2$. Now if $d_{n-1} > 0$, the line bundle $U_{n-1}^2 K$ has no base points (by Riemann–Roch) and so the isomorphism ϕ_0 in (9.14) must come from an isomorphism of line bundles $U_{n-2} U_{n-1} \cong U_{n-1}^2 K$, so $U_{n-2} \cong U_{n-1} K$. On the other hand, if $d_{n-1} < 0$ then the degree of $U_{n-1}^2 \cong U_{n-1} U_n K$ (by (9.15)) is negative and so $H^1(\Sigma; U_{n-1} U_n K)$ has dimension given by Riemann–Roch. Since $U_{n-2} U_n$ then also has negative degree, the isomorphism ϕ_1 implies $\deg U_{n-2} = \deg U_{n-1} K$ and so $\Phi: U_{n-2} \to U_{n-1} K$ is an isomorphism in this case too. There remains the situation $d_{n-1} = 0$. Now if U_{n-1}^2 is trivial, $U_{n-1}^2 K \cong K$ has no base points, so we again get an isomorphism $U_{n-2} \cong U_{n-1} K$ from (9.14). If U_{n-1}^2 is non-trivial, then $\dim H^1(\Sigma; U_{n-1}^2) = g - 1$ and if $\deg U_{n-2} U_n < \deg U_{n-1}^2 = 0$ then Riemann–Roch gives $\dim H^1(\Sigma; U_{n-2} U_n) = -d_{n-2} - d_n + g - 1$. Thus if ϕ_1 is an isomorphism, then $d_n + d_{n-2} = 0$. Together with $d_{n-1} = 0$ this gives $d_{n-2} + d_{n-1} + d_n = 0$ which contradicts the stability condition for the Φ-invariant subbundle $U_{n-2} \oplus U_{n-1} \oplus U_n$. In all cases then $\deg U_{n-2} = \deg U_{n-1}K$ and so Φ defines an isomorphism

$$U_{n-2} \cong U_{n-1}K \cong U_n K^2.$$

After having seen the pattern and techniques, we can now start the induction to prove the lemma.

Suppose inductively that $U_m, \ldots, U_n$ are line bundles and for all $k \geq m$,

$$U_k \cong U_n K^{n-k}. \tag{9.16}$$

We have seen above that this is true for $m = n, n-1$ and $n-2$, so we may as well assume that $m < n-1$. Now in particular the degree of U_k is given from (9.16) by

$$d_k = d_n + 2(n-k)(g-1). \tag{9.17}$$

Assume now that $\operatorname{rk} U_{m-1} > 1$, then the Riemann–Roch inequality (9.11) gives

$$\dim H^1(\Sigma; W_{m-1} \otimes U_n) \geq g - d_n.$$

But

$$\begin{aligned} \deg L_{m-1} U_n &\leq \deg U_m U_n K \\ &= 2d_n + 2(n-m+1)(g-1) \qquad \text{from (9.17)} \end{aligned}$$

Hence, since B in (9.9) is surjective we must have the inequality:

$$2d_n + (2n - 2m + 1)(g-1) \geq g - d_n$$

or equivalently

$$3d_n + 2(n-m)(g-1) \geq 1. \tag{9.18}$$

But now stability of the Φ-invariant sub-bundle $U_m \oplus \cdots \oplus U_n$ implies

$$0 > d_m + \cdots + d_n = (n-m+1)d_n + (n-m)(n-m+1)(g-1)$$

where we have used (9.17).

Thus $d_n < -(n-m)(g-1)$ and substituting in (9.18) this gives

$$-(n-m)(g-1) > 1$$

which is a clear contradiction.

We deduce by induction that if $m > 1/2$ and $U_m, \ldots, U_n$ are line bundles, then so is U_{m-1}.

Now we need to show that $U_{m-1} \cong U_m K$, and for this we consider first (9.14) with $m+1$ replaced by m. Using the hypothesis $U_k \cong U_n K^{n-k}$, we have an isomorphism

$$H^0(\Sigma; U_{m-1} U_n K^{n-m}) \to H^0(\Sigma; U_n^2 K^{2n-2m+1}).$$

If $d_n > -2(n-m)(g-1)$ the line bundle $U_n^2 K^{2n-2m+1}$ has no base points in which case the degrees of U_{m-1} and $U_n K^{n-m+1}$ are the same and so the homomorphism Φ between them is an isomorphism. Similarly, considering (9.13) with $m+1$ replaced by m, we have an isomorphism

$$H^1(\Sigma; U_{m-1} U_n) \to H^1(\Sigma; U_n^2 K^{n-m+1}).$$

If $d_n < -(n-m+1)(g-1)$ then Riemann–Roch gives the dimensions of both spaces, and so again $U_{m-1} \cong U_n K^{n-m}$. If neither of these inequalities holds, then

$-(n-m+1)(g-1) \le d_n \le -2(n-m)(g-1)$ which yields

$$(n-m-1)(g-1) \le 0$$

which for $m < n-1$ is a contradiction. We have thus completed the induction and U_m is a line bundle for $m \ge 0$. Using the duality $U_{-m} \cong U_m^*$ we have the result for all m and so have proved the lemma.

Note now that the proof of the lemma gives more information, namely if $n > 1/2$ then $U_m \cong U_n K^{n-m}$, the isomorphism being given by the Higgs field Φ. If V is of odd rank $2n+1$ there is an odd number of U_i's and in particular a bundle U_0. Since $U_0 \cong U_0^*$, then U_0^2 is trivial and $U_0 \cong U_n K^n$. Hence,

$$V \cong U_0 \otimes \left\{ \bigoplus_{m=-n}^{n} K^m \right\} = U_0 \otimes S^{2n}(K^{-1/2} \oplus K^{1/2}).$$

Projectively, this is equivalent to the basic Higgs bundle in §3.

Similarly, if V has even rank $2n+2$, $U_{1/2}^* \cong U_{-1/2} \cong U_{1/2}K$ and so

$$V \cong S^{2n+1}(K^{-1/2} \oplus K^{1/2}).$$

We thus obtain the following proposition:

PROPOSITION 9.19. *Let (V, Φ) be a* **stable** *local minimum of f on $\mathbf{M}_0$, then (V, Φ) is isomorphic to one of the following:*

(1) *a stable bundle V with $\Phi = 0$*
(2) *a rank 2 Higgs bundle of the form $V = L \oplus L^*$, with Φ a non-zero section of $L^{-2}K$*
(3) *a Higgs bundle of the form $V = S^n(K^{-1/2} \oplus K^{1/2})$ with Φ: $K^{m+1} \to K^m \otimes K$ the identity.*

All we need note for the proof here is that the isomorphism $U_{n-1} \cong U_n K$ required $n > 1/2$ and therefore is not valid in rank 2. In fact the Higgs bundles of type (2) lie in different components depending on the degree of L and are actually minima of f (see [6]).

§10. GENERAL MINIMA

So far we have found candidates for the minima of $f = i\int_\Sigma \operatorname{tr} \Phi\Phi^*$ which are stable (or irreducible) and correspond to smooth points of the moduli space. It remains to eliminate the non-trivial reducible ones. Suppose then that (V, Φ) is a direct sum of two irreducible Higgs bundles $(V_1 \oplus V_2, \Phi_1 \oplus \Phi_2)$ and is a minimum for f. Varying each individually shows that (V_1, Φ_1) and (V_2, Φ_2) must be one of the three types of Proposition (9.19). If they are both of type (1), then since $f = 0$, we are clearly at a minimum, so let us first consider (V_1, Φ_1) of type (1) and (V_2, Φ_2) of type (2).

Thus V_1 is a stable bundle (with a flat $SO(n)$-connection) and $V_2 = L \oplus L^*$ with $\deg L \ge 1$ by the stability of (V_2, Φ_2) since L^* is Φ_2-invariant. Consider the space

$$H^1(\Sigma; V_1 \otimes L^*) \subseteq H^1(\Sigma; V_1 \otimes V_2) \subseteq H^1(\Sigma; \Lambda^2 V).$$

Since V_1 has zero weight and L^* weight 1/2, this subspace has positive weight. Moreover, $\Phi(V_1) = 0$ since $\Phi_1 = 0$ and $\Phi(L^*) = 0$ since $\Phi_2(L^*) = 0$. The subspace is also nontrivial since a standard vanishing theorem using the flat connection on V_1 shows that $H^0(\Sigma; V_1 \otimes L^*) = 0$ and hence by Riemann–Roch,

$$\begin{aligned}\dim H^1(\Sigma; V_1 \otimes L^*) &= -\deg V_1 \otimes L^* + (g-1)\operatorname{rk} V_1 \\ &= (g-1+\deg L)\operatorname{rk} V_1 > 0.\end{aligned}$$

We therefore obtain a contradiction to minimality so long as we can prove that this infinitesimal deformation can be integrated to a one-parameter family.

To do this, let us consider how a class $[\alpha] \in H^1(\Sigma; V_1 \otimes L^*)$ can be used to construct a Higgs bundle. The obvious way to construct a vector bundle from such a cohomology class is as an extension:

$$0 \to V_1 \to W_1 \to L \to 0.$$

However, if V_1 is an orthogonal bundle, we can construct also an orthogonal bundle of rank two greater than rk V_1. In terms of transition functions, we take

$$\begin{pmatrix} l^{-1} & \alpha & 0 \\ 0 & G & -\alpha \\ 0 & 0 & l \end{pmatrix}$$

where G is a transition matrix for V_1 which preserves the quadratic form, l and l^{-1} are transition functions for L and L^* respectively, and α is a Čech representative for $[\alpha] \in H^1(\Sigma; V_1 \otimes L^*)$. This defines a bundle W with a quadratic form, a projection to L and an inclusion $L^* \subset W$ whose quotient is the extension W_1 above.

We canonically define an associated Higgs field by:

$$\Phi\colon W \to L \xrightarrow{\Phi_2} L^* K \to W \otimes K.$$

Multiplying α by a scalar λ provides the one-parameter family of deformations of the Higgs bundle in the direction $[\alpha]$. We deduce therefore from the positive weighting, that a bundle of this type cannot be a minimum for f.

The other cases can be dealt with in a similar manner, taking the highest weight eigenspace of ψ from each factor. Thus, if $V_1 = \bigoplus_{m=-n}^{n} U_m$ and $V_2 = \bigoplus_{r=-s}^{s} W_r$ then we take a class in $H^1(\Sigma; U_n \otimes W_s)$. This is annihilated by Φ and is contained in $H^1(\Sigma; \Lambda^2 V)$—it is non-trivial by stability and degree considerations. If $U_n \neq U_{-n}$ and $W_s \neq W_{-s}$ we form the extension

$$0 \to U_n \xrightarrow{i} E \xrightarrow{p} W_{-s} \to 0$$

from this class and define

$$V = \bigoplus_{m=-n+1}^{n-1} U_m \oplus E \oplus \bigoplus_{r=-s+1}^{s-1} W_r \oplus E^*.$$

The corresponding Higgs field is given by

$$\begin{aligned}\Phi(&u_{-n+1}, \ldots, u_{n-1}, e, w_{-s+1}, \ldots, w_{s-1}, \tilde{e}) \\ &= (i^*\tilde{e}, \Phi_1 u_{-n+1}, \ldots, \Phi_1 u_{n-2}, i\Phi_1 u_{n-1}, pe, \Phi_2 w_{-s+1}, \ldots, \Phi_2 w_{s-2}, p^*\Phi_2 w_{s-1}).\end{aligned}$$

If $U_n = U_{-n}$ then U is of type (1) and the construction is just like the first case considered above. In each case, unless $\Phi = 0$, we obtain positive weights and hence directions in which the function f has negative second variation. Thus we obtain:

Proposition (10.1). *Let (V, Φ) be a local minimum of f on M_0 with* rk $V > 2$, *then either $\Phi = 0$ or (V, Φ) is equivalent to a Higgs bundle $V = S^n(K^{-1/2} \oplus K^{1/2})$ with $\Phi\colon K^{m+1} \to K^m \otimes K$ the identity.*

Now consider the general case of a flat $PSL(n, \mathbb{R})$-bundle, which does not lift to an $SL(n, \mathbb{R})$-bundle. We need to modify the vector bundle V to have a quadratic form Q with values in a line bundle L of degree 1:

$$Q\colon V \otimes V \to L$$

In particular, $V^* \cong V \otimes L^*$. The structure of the arguments in §9 and above remains unchanged but the inequalities in Lemma (9.6) are changed slightly. This necessitates occasionally dealing with the special case $g = 2$ in isolation, but the result remains the same. We leave the details to the reader. The conclusion is that the corresponding moduli spaces for $n > 2$ are connected, since our Teichmüller components consist of bundles which lift to $SL(n, \mathbb{R})$.

Putting these results together we obtain the following:

THEOREM (10.2). *Let Σ be a compact oriented surface of genus $g > 1$ with fundamental group $\pi_1(\Sigma)$. Denote by* $\mathrm{Hom}^+(\pi_1(\Sigma); PSL(n, \mathbb{R}))$ *the space of completely reducible homomorphisms from $\pi_1(\Sigma)$ to $PSL(n, \mathbb{R})$, and $\mathbf{M}^+$ the quotient space by the conjugation action of $PSL(n, \mathbb{R})$. Suppose $n > 2$. Then if n is odd, $\mathbf{M}^+$ has 3 connected components and if n is even $\mathbf{M}^+$ has 6 components. In the first case one of the components is diffeomorphic to $\mathbb{R}^{(n^2-1)(g-1)}$, in the second case two.*

(Note that, for purposes of comparison, the corresponding result for $n = 2$ ([4], [6]) is that $\mathbf{M}^+$ has $4g - 3$ components, two of which are diffeomorphic to $\mathbb{R}^{6g-6}$).

Proof. By the properness of f, each component must contain a minimum. But Proposition (10.1) shows that the only minima are either flat $PSO(n)$-connections (the case of $\Phi = 0$) or a connection which we showed in (7.5) to lie in the Teichmüller component. Certainly since $\Phi \neq 0$ for any Higgs bundle in the Teichmüller component these are disjoint possibilities. On the other hand, we know from [1] that the moduli space of flat $PSO(n)$-connections on a fixed bundle is connected. Thus it is only the topological type of the underlying bundle which distinguishes the components containing compact group connections.

The topological type of those bundles which give the Teichmüller component can easily be determined. Firstly, they arise from flat connections on a *vector* bundle, so we are considering homomorphisms from $\pi_1(\Sigma)$ to $PSL(n, \mathbb{R})$ which lift to $SL(n, \mathbb{R})$. Beyond this, for $n > 2$, it is the second Stiefel–Whitney class $w_2(E)$ of the associated rank n real vector bundle E which defines the topological equivalence class.

In the odd case $n = 2m + 1$, the complexification of E is

$$V = \bigoplus_{l=-m}^{m} K^l$$

and the real structure (as described in §6) gives

$$E \cong 1 \oplus \bigoplus_{l=1}^{m} K^l$$

as a *real* vector bundle. Thus $w_2(E)$ is the mod 2 reduction of the first Chern class of this complex vector bundle. This is zero since $c_1(K^l) = 2l(g-1) = 0 \bmod 2$.

In the even case $n = 2m$, we have

$$E \cong \bigoplus_{l=1}^{m} K^{l-1/2}$$

as a real vector bundle, so

$$w_2(E) = \sum_{l=1}^{m} (g-1)(2l-1) = (g-1)m^2 \bmod 2.$$

Each moduli space of flat connections on a fixed bundle is therefore connected except for the case $w_2(E) = 0$ for n odd and $w_2(E) = (g-1)m^2$ for $n = 2m$, where the Teichmüller component appears as well as the component containing compact group connections.

The final remark to complete the component count is to note that $\mathbf{M}$, where the analysis was carried out, consists of equivalence classes of representations in $PSL(n, \mathbb{C})$ modulo the conjugation action of that group. But now $PSL(n, \mathbb{C}) = PGL(n, \mathbb{C})$ but $PSL(n, \mathbb{R}) \neq PGL(n, \mathbb{R})$ if n is even. In this case the subgroup of $PSL(n, \mathbb{C})$ which takes $PSL(n, \mathbb{R})$ to itself by conjugation is the 2-component group $PGL(n, \mathbb{R})$. Thus in the moduli space of flat $PSL(n, \mathbb{R})$-connections with n even, the contractible Teichmüller component appears twice.

The description of components thus proceeds as follows:

(1) If n is odd, $PSL(n, \mathbb{R}) = SL(n, \mathbb{R})$ and if $w_2(E) = 0$ there are two components, if $w_2(E) \neq 0$ just one.

(2) If $n = 2m$, there are four topological types (8.1). Two do not lift to $SL(n, \mathbb{R})$ and so give connected spaces. Two do lift and if $w_2(E) \neq (g-1)m^2$ this gives a connected space. If $w_2(E) = (g-1)m^2$ we have three components—the one containing $PSO(n)$-connections and the Teichmüller component appearing twice.

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